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Research Article

Continuous Dependence in Front Propagation for Convective Reaction-Diffusion Models with Aggregative Movements

Luisa Malaguti,[1] Cristina Marcelli,[2] and Serena Matucci[3]

[1] *Department of Engineering Sciences and Methods, University of Modena and Reggio Emilia, 42122 Reggio Emilia, Italy*
[2] *Department of Mathematical Sciences, Polytechnic University of Marche, 60131 Ancona, Italy*
[3] *Department of Electronic and Telecommunications, University of Florence, 50139 Florence, Italy*

Correspondence should be addressed to Cristina Marcelli, c.marcelli@univpm.it

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The paper deals with a degenerate reaction-diffusion equation, including aggregative movements and convective terms. The model also incorporates a real parameter causing the change from a purely diffusive to a diffusive-aggregative and to a purely aggregative regime. Existence and qualitative properties of traveling wave solutions are investigated, and estimates of their threshold speeds are furnished. Further, the continuous dependence of the threshold wave speed and of the wave profiles on a real parameter is studied, both when the process maintains its diffusion-aggregation nature and when it switches from it to another regime.

1. Introduction

This paper deals with the reaction-diffusion equation:

$$v_t + h(v)v_x = (D(v)v_x)_x + f(v), \quad t \geq 0, \; x \in \mathbb{R}, \tag{1.1}$$

which provides an interesting model in several frameworks such as the population dispersal, ecology, nerve pulses, chemical processes, epidemiology, cancer growth, chemotaxis processes etc. The great variety of contexts where it meets justifies its interest by the scientific community. We refer to the books [1–3] for an updated illustration of the main properties and applications of (1.1).

In many cases the presence of a convective flux, expressed by the function h, is a fundamental ingredient of these models. Indeed, it accounts for external forces acting into the process such as, for instance, control strategies. We assume that f is a monostable, that is, of Fisher-type, reaction term; that is, it satisfies

$$f(u) > 0 \quad \text{for } u \in (0,1), \qquad f(0) = f(1) = 0. \tag{1.2}$$

According to (1.2), $v \equiv 0$ and $v \equiv 1$ are stationary solutions of (1.1). Since v frequently represents a density, of main interest is the investigation of solutions taking values in the interval $[0,1]$ and connecting these two stationary states. The family of traveling wave solutions (t.w.s.) is an important class of functions which share this property. It was showed in fact that, at least in some sense, t.w.s. are able to capture the main features of all the solutions taking values in $[0,1]$ (see, e.g., [4, 5]). We recall that a function $v = v(t,x)$ is said to be a t.w.s. of (1.1) with wave speed c if there exist a maximal interval (a,b), with $-\infty \le a < b \le +\infty$, and a function $u \in C(cl(a,b)) \cap C^2(a,b)$, satisfying

$$\lim_{\xi \to a^+} D(u(\xi))u'(\xi) = \lim_{\xi \to b^-} D(u(\xi))u'(\xi) = 0, \tag{1.3}$$

such that $v(t,x) = u(x - ct)$ is a solution of (1.1) for all $t \in (a,b)$. The symbol $cl(a,b)$ denotes the closure in $\mathbb{R}$ of (a,b), and the quantity $\xi = x - ct$ is usually called the wave coordinate.

Since we restrict our analysis to those solutions of (1.1) with values in $[0,1]$, we assume f, h, D defined in such an interval and, throughout the paper, we take $h, f \in C[0,1]$ and $D \in C^1[0,1]$.

A population in a hostile environment usually increases its chances to survive if it is able to produce aggregative movements. Therefore, when (1.1) is a model for population dynamics, it has to take into account also this important aspect. As a prototype of equation (1.1) in this framework, we can consider

$$v_t = -\left[v k_0 \left(1 - \frac{v}{w}\right) v_x\right]_x + \mu v_{xx} + f(v), \tag{1.4}$$

where μ, k_0, and w denote positive constants while the function f accounts for the net rate of growth of the individuals and it satisfies (1.2). In this case $D(v) = \mu - v k_0 (1 - v/w)$, and (1.4) is a generalized version of a model proposed by Turchin [6] in order to describe the aggregative movements of *Aphis varians*. Under suitable conditions on the parameters, it is straightforward to show the existence of a value $v_0 \in (0,1)$ such that $D(v) > 0$ when $v \in [0, v_0)$ while $D(v) < 0$ in $(v_0, 1]$. This interesting situation is motivated by the remark that it is highly unlikely to find conspecific in the vicinity at low population densities. It seems then more reasonable to expect that the tendency of a population to aggregate, modeled by negative values of D, appears after a certain threshold density level. Therefore, in order to include this type of aggregative behavior into the model, we assume the existence of $\alpha \in [0,1]$ satisfying

$$D(u) > 0 \quad \text{for } u \in (0,\alpha), \quad D(u) < 0, \quad \text{for } u \in (\alpha, 1). \tag{1.5}$$

The extremal cases $\alpha = 1$ and $\alpha = 0$ respectively correspond to a purely diffusive and a purely aggregative term. In the former case, that is, $\alpha = 1$, the presence of t.w.s. in these models and their main qualitative properties have been investigated since a long time, and we refer to [2, 7, 8] for details. The latter case; that is, $\alpha = 0$, was recently discussed in [9]. As far as we know, the first detailed investigation concerning the existence of t.w.s. of (1.1) when D satisfies (1.5) and $\alpha \in (0,1)$ appeared in [10], but there was no convective effect included. Under the additional assumption that

$$\dot{D}(\alpha) < 0, \tag{1.6}$$

we complete in Theorem 2.1 the analysis started in [10] and show that also (1.1), incorporating a convective behavior, is able to support a continuum of t.w.s., parameterized by their wave speeds c which satisfy $c \geq c^*$, and we give an estimate of the threshold speed c^*.

The very nature of aggregative processes causes ill posedness of our model (1.1) when $\alpha \in (0,1)$ and $v \in [\alpha, 1]$ (see, e.g., [10] and the references there contained). However, discrete models underlying (1.1) are well posed, and some numerical computations (see [6]) seem in good agreement with the information obtained in the discrete setting. Moreover, even in this possible ill-posedness context, the t.w.s. above defined are *regular* solutions for (1.1), and this increases the interest in studying them.

As it is clear also from the prototype equation (1.4), very naturally these models include real parameters which frequently cause, on their varying, the transition of the process a diffusive to a diffusive-aggregative regime or from a diffusive-aggregative to a purely aggregative one. The main aim of this paper lies in the analysis of these behaviors, so we consider the more general dynamic:

$$v_t + h(v;k)v_x = (D(v;k)v_x)_x + f(v;k), \quad t \geq 0, \ x \in \mathbb{R}, \tag{1.7}$$

which continuously depends on the real parameter $k \in [0,1]$. We take h, D, and f in $C([0,1]^2)$ with $D(\cdot;k) \in C^1[0,1]$ and

$$f(u;k) > 0 \quad \text{for } u \in (0,1), \qquad f(0;k) = f(1;k) = 0, \tag{1.8}$$

for all k. Further, we assume the existence of a continuous function $\alpha : [0,1] \to [0,1]$ such that

$$\alpha(1) = 0, \quad \alpha(0) = 1, \qquad \alpha(k) \in (0,1), \qquad \text{for } 0 < k < 1, \tag{1.9}$$

$$D(u;k) > 0 \quad \text{for } u \in (0, \alpha(k)), \quad D(u;k) < 0, \quad \text{for } u \in (\alpha(k), 1), \tag{1.10}$$

$$\dot{D}(\alpha(k);k) < 0, \quad \text{for } k \in (0,1). \tag{1.11}$$

We discuss both the case $k \in (0,1)$, corresponding to a diffusive-aggregative dynamic, as well as the case $k = 0$ or 1, where (1.7) is, respectively, a purely diffusive reaction-convection equation and a purely aggregative one. In Theorem 2.2 we prove that the threshold value $c^*(k)$ is always a lower semicontinuous function (l.s.c.) on the whole interval $[0,1]$; hence, in particular, the lower semicontinuity holds when the process (1.7) switches from a purely diffusive to a diffusive-aggregative and from the latter to a purely aggregative behavior. We provide quite general conditions (see Theorem 2.2 and Proposition 2.4) either guaranteeing that $c^*(k)$ is continuous on $[0,1]$ or that it fails to be continuous for $k = 0$ or $k = 1$.

Two simple illustrative examples complete this discussion. In Theorem 2.6 we show the continuous dependence, on the parameter k, of any family of profiles $u(\xi;k)$ corresponding to a continuum of wave speeds $\gamma(k)$. These are the main results of the paper; their statements appear in Section 2 while their proofs can be found in Section 4. Notice that the function α can be extended in a continuous way outside the interval $[0,1]$, so that the model also includes the purely diffusive or aggregative dynamic. However, the study of the continuity in the former case has been carried on in [11], while the latter case can be treated by a suitable change of variable, and this discussion is included in Theorem 4.2.

As for the methodology, it is easy to see that every wave profile $u(\xi)$ of (1.1), having wave speed c, corresponds to a *regular* solution of the boundary value problem:

$$\begin{gathered}(D(u)u')' + (c - h(u))u' + f(u) = 0,\\ u(a^+) = 1, \qquad u(b^-) = 0,\end{gathered} \tag{1.12}$$

where $-\infty \le a < b \le \infty$. More precisely, a solution of (1.12) in (a,b) is a function $u \in C(cl(a,b)) \cap C^2(a,b)$ satisfying the boundary conditions (1.3).

Since, in addition, u is strictly monotone in every interval where $0 < u(\xi) < 1$ (see, e.g., Theorem 3.7), the investigation of t.w.s. can be reduced to the study of a first-order singular boundary value problem. In order to investigate this problem, we mainly use comparison-type techniques, that is, suitable upper and lower solutions. In Section 3 we report and complete the original discussion developed in [8] and show the equivalence between the first-order singular problem and the existence of t.w.s. for (1.1).

2. Statements of the Main Results

Here we present the main results of the paper, which will be proved in Section 4. Put $\varphi(u) := D(u)f(u)$; in the whole section we assume that

$$\varphi \text{ is differentiable at } u = 0 \text{ and } 1. \tag{2.1}$$

The first result concerns the existence of t.w.s. for (1.1).

Theorem 2.1. *Assume that conditions* (1.2), (1.5), (1.6) *and* (2.1) *hold. Then, there exists a value c^* such that* (1.1) *supports t.w.s. if and only if $c \ge c^*$, and the profile is unique, up to shifts. Moreover, the threshold value c^* satisfies the following estimates, depending on the value α:*

$$h(0) + 2\sqrt{\dot{\varphi}(0)} \le c^* \le \max_{u\in[0,1]} h(u) + 2\sqrt{\sup_{u\in(0,1]} \frac{\varphi(u)}{u}}, \quad \text{if } \alpha = 1, \tag{2.2}$$

$$h(1) + 2\sqrt{\dot{\varphi}(1)} \le c^* \le \max_{u\in[0,1]} h(u) + 2\sqrt{\sup_{u\in[0,1)} \frac{\varphi(u)}{u-1}}, \quad \text{if } \alpha = 0, \tag{2.3}$$

and, if $0 < \alpha < 1$,

$$\max\left\{2\sqrt{\dot{\varphi}(0)}+h(0),2\sqrt{\dot{\varphi}(1)}+h(1)\right\}\le c^*$$
$$\le \max\left\{2\sqrt{\sup_{u\in(0,\alpha]}\frac{\varphi(u)}{u}}+\max_{u\in[0,\alpha]}h(u),2\sqrt{\sup_{u\in[\alpha,1)}\frac{\varphi(u)}{u-1}}+\max_{u\in[\alpha,1]}h(u)\right\}. \tag{2.4}$$

As stated in the Introduction section, the main aim of the paper is the study of the continuous dependence, both of the threshold value c^* and of the wave profiles u, with respect to the coefficients h, D, and f appearing in (1.1). More precisely, we are mainly interested in studying the continuous dependence when a change in the type of dynamics occurs: from a purely diffusive to a diffusive-aggregative one or from a diffusive-aggregative to a purely aggregative one.

To this aim, we consider a continuous function $\alpha : [0,1] \to [0,1]$ satisfying conditions (1.9), and we introduce a real parameter k varying in $[0,1]$, inside the coefficients of (1.1), in such a way that $h(u;k)$, $D(u;k)$, and $f(u;k)$ are continuous functions of two variables. We further assume (1.8), (1.10), and (1.11) and suppose that $\varphi(u;\cdot)$ satisfy (2.1) for each $k\in[0,1]$.

In this framework, the threshold value c^* is a function of the parameter k, say $c^* = c^*(k)$, and, for every $c \ge c^*(k)$, the profile u (modulo shifts) of the t.w.s. having speed c is a function of (ξ,k), where ξ is the wave variable, say $u=u(\xi;k)$.

As for the function $c^*(k)$, the following result will be proven.

Theorem 2.2. *Let $\alpha : [0,1] \to [0,1]$ and $h,D,f : [0,1]\times[0,1] \to \mathbb{R}$ be continuous functions satisfying* (1.8), (1.9), (1.10), *and* (1.11). *For all $k\in[0,1]$, assume that condition* (2.1) *holds for $\varphi(\cdot;k)$, and let $D(\cdot;k)\in C^1([0,1])$. Then, $c^*(k)$ is l.s.c..*

Further, if for some $k_0\in(0,1)$ we have

$$\limsup_{(u,k)\to(0,k_0)}\frac{\varphi(u;k)-\varphi(u;k_0)}{u}\le 0,\qquad \liminf_{(u,k)\to(1,k_0)}\frac{\varphi(u;k)-\varphi(u;k_0)}{1-u}\ge 0, \tag{2.5}$$

then c^* is continuous at k_0.

With regards to the behavior of c^* at the endpoints $k=0$ and $k=1$, if

$$\limsup_{(u,k)\to(1,0)}\left\{\frac{\varphi(u;k)}{u-1}-\dot{\varphi}(1;k)\right\}\le 0, \tag{2.6}$$

$$2\sqrt{\limsup_{k\to 0}\dot{\varphi}(1;k)}+h(1;0)\le 2\sqrt{\dot{\varphi}(0;0)}+h(0;0) \tag{2.7}$$

then $c^*(k)$ is continuous at $k=0$, and, similarly, if

$$\limsup_{(u,k)\to(0,1)}\left\{\frac{\varphi(u;k)}{u}-\dot{\varphi}(0;k)\right\}\le 0, \tag{2.8}$$

$$2\sqrt{\limsup_{k\to 1}\dot{\varphi}(0;k)}+h(0;1)\le 2\sqrt{\dot{\varphi}(1;1)}+h(1;1) \tag{2.9}$$

then $c^*(k)$ is continuous at $k=1$.

Remark 2.3. Assume now that $\dot{\varphi}(u;k)$ exists and it is continuous on $[0,1] \times [0,1]$. In this case it is easy to verify that all conditions in (2.5), (2.6), and (2.8) are satisfied as equalities. Furthermore, according to (1.8) and (1.10), we obtain that $\dot{\varphi}(1;0) \le 0$ and also $\dot{\varphi}(0;1) \le 0$ while $\dot{\varphi}(1;k) \ge 0$ and $\dot{\varphi}(0;k) \ge 0$ for $k \in (0,1)$. The continuity of $\dot{\varphi}$, hence, implies that $\dot{\varphi}(1;0) = \dot{\varphi}(0;1) = 0$. Consequently, conditions (2.7) and (2.9), respectively, reduce to

$$h(1;0) \le 2\sqrt{\dot{\varphi}(0;0)} + h(0;0), \tag{2.10}$$

$$h(0;1) \le 2\sqrt{\dot{\varphi}(1;1)} + h(1;1). \tag{2.11}$$

Therefore, under conditions (2.10) and (2.11), the function $c^*(k)$ is continuous on the whole interval $[0,1]$. Of course, when $h \equiv 0$, then (2.10) and (2.11) trivially hold, and $c^*(k)$ is continuous on $[0,1]$.

As an immediate consequence of the estimates (2.2), (2.3), and (2.4), if the relation (2.7) or (2.9) in the previous theorem does not hold, then the function $c^*(k)$ could be discontinuous at the endpoints, as the following result states.

Proposition 2.4. *Under the same conditions of Theorem 2.2, if*

$$2\sqrt{\limsup_{k\to 0} \dot{\varphi}(1;k)} + h(1;0) > 2\sqrt{\sup_{u\in(0,1]} \frac{\varphi(u;0)}{u}} + \max_{u\in[0,1]} h(u;0), \tag{2.12}$$

then $c^*(k)$ is not continuous at $k = 0$. Similarly, if

$$2\sqrt{\limsup_{k\to 1} \dot{\varphi}(0;k)} + h(0;1) > 2\sqrt{\sup_{u\in[0,1)} \frac{\varphi(u;1)}{u-1}} + \max_{u\in[0,1]} h(u;1), \tag{2.13}$$

then $c^*(k)$ is not continuous at $k = 1$.

Example 2.5. Let $\alpha(k) := 1 - k$ and $D(u;k) := \alpha(k) - u = 1 - k - u$, and let

$$f(u;k) := \begin{cases} u\sqrt{1-u}, & \text{if } k = 0, \\ \min\left\{u\sqrt{1-u},\ \dfrac{2}{k}(1-u)\right\}, & \text{if } k \ne 0. \end{cases} \tag{2.14}$$

Observe that conditions (1.8), (1.9), and (1.10) are trivially satisfied. Moreover, $f(u;k)$ uniformly converges to $f(u;0)$ as $k \to 0$, and f is continuous in $[0,1] \times [0,1]$. Furthermore, also condition (2.1) is satisfied. Indeed, $\varphi(u;0) = u(1-u)^{3/2}$ is differentiable in $[0,1]$, $\dot{\varphi}(0;k) = (1-k)$ and $\dot{\varphi}(1;k) = 2$ for every $k \in (0,1]$, and finally $\varphi(u;k)$ is differentiable at

$u = \alpha(k)$ since for $u = \alpha(k) = 1 - k$ we have $u\sqrt{1-u} < (2/k)(1-u)$, and then $f(u;k)$ is differentiable at $u = \alpha(k)$. Since

$$\lim_{u\to 1}\frac{\varphi(u;k)}{u-1} = \lim_{u\to 1}\frac{1-k-u}{u-1}\frac{2(1-u)}{k} = 2, \quad \text{for every } k \neq 0, \tag{2.15}$$

we have $\lim_{k\to 0}\dot{\varphi}(1;k) = 2$, whereas

$$\sup_{u\in(0,1]}\frac{\varphi(u;0)}{u} = \sup_{u\in(0,1]}(1-u)^{3/2} = 1, \tag{2.16}$$

then (2.12) holds for every continuous function $h : [0,1]\times[0,1] \to \mathbb{R}$ such that $h(1;0) = \max_{[0,1]} h(u;0)$. Therefore, the function c^* is not continuous at $k = 0$; that is, c^* looses its continuity in the transition from a diffusive-aggregative dynamic to a purely diffusive one.

The last result concerns the continuous dependence of the profiles of the t.w.s..

Theorem 2.6. *Let $\alpha : [0,1] \to [0,1]$ and $h, D, f : [0,1]\times[0,1] \to \mathbb{R}$ be continuous functions satisfying* (1.8), (1.9), (1.10), *and* (1.11). *For all $k \in [0,1]$, assume that condition* (2.1) *holds for $\varphi(\cdot;k)$, and let $D(\cdot;k) \in C^1([0,1])$. Take $\gamma : [0,1] \to [0,1]$ continuous and such that $\gamma(k) \geq c^*(k)$ for every $k \in [0,1]$, and let $k_0 \in [0,1]$ be fixed.*

Let $U_k(\xi) := U(\xi;\gamma(k),k)$ be the profile of the corresponding travelling wave solution of (1.1) *with speed $\gamma(k)$ such that $U_k(0) = u_0$, for some fixed $u_0 \in (0,1)$. Then $U_k(\xi)$ converges to $U_{k_0}(\xi)$ uniformly on all the real line.*

Remark 2.7. In all the results stated in this section, if convective effects are not present; that is, $h(u;k) \equiv 0$, then assumptions (1.6) and (1.11) can be removed, as we show in the end of the proof of Theorem 3.7.

3. Reduction to a Singular First-Order Equation

Given continuous functions $h, D, f : [0,1] \to [0,1]$, with $D \in C^1[0,1]$, satisfying conditions (1.2), (1.5), (1.6), and (2.1), let us consider the following singular first-order boundary value problem

$$\begin{gathered}\dot{z}(u) = h(u) - c - \frac{\varphi(u)}{z(u)}, \quad \text{for every } u \in (0,1)\setminus\{\alpha\},\\ z(0^+) = z(1^-) = 0,\\ z(u)\varphi(u) < 0, \quad \forall\, u \in (0,1),\ u \neq \alpha,\end{gathered} \tag{3.1}$$

where c is an unknown real constant. By a solution of the b.v.p. (3.1), we mean a differentiable function $z : (0,1) \to \mathbb{R}$, satisfying all the conditions in (3.1). Of course, any possible solution of (3.1) is of class C^1 in any compact interval $[a,b] \subset (0,1)$ not containing the value α.

As we will show at the end of this section, the solvability of problem (3.1) is equivalent to the existence of solutions of (1.12). For this reason, we now study the existence of solutions to problem (3.1). To this aim, in [12] it was proved the existence of solutions to the singular boundary value problem (3.1) in the case when φ is everywhere positive. In particular, the following Proposition is a consequence of [12, Lemma 2.2] combined to [12, Theorem 1.4].

Proposition 3.1. *Let $h, \varphi : [\ell_1, \ell_2] \to \mathbb{R}$ be continuous functions such that $\varphi(\ell_1) = \varphi(\ell_2) = 0$, $\varphi(u) > 0$ in (ℓ_1, ℓ_2), and $\dot{\varphi}(\ell_1)$ exists finite. Then, there exists a value c^*, satisfying the following estimate:*

$$2\sqrt{\dot{\varphi}(\ell_1)} + h(\ell_1) \le c^* \le 2\sqrt{\sup_{u\in(\ell_1,\ell_2]} \frac{\varphi(u)}{u-\ell_1}} + \max_{u\in[\ell_1,\ell_2]} h(u), \tag{3.2}$$

such that the problem

$$\begin{gathered} \dot{z}(u) = h(u) - c - \frac{\varphi(u)}{z(u)}, \quad \text{for every } u \in (\ell_1, \ell_2), \\ z(\ell_1^+) = z(\ell_2^-) = 0, \\ z(u)\varphi(u) < 0, \quad \forall\, u \in (\ell_1, \ell_2), \end{gathered} \tag{3.3}$$

admits solutions if and only if $c \ge c^*$. Moreover, the solution is unique.

As far as the regularity of the solution z at the endpoints, the following result holds.

Corollary 3.2. *Under the same assumptions of Proposition 3.1, if $\dot{\varphi}(\ell_2)$ exists finite, then $\dot{z}(\ell_2)$ exists finite too, and*

$$\dot{z}(\ell_2) = \frac{1}{2}\left(h(\ell_2) - c + \sqrt{(h(\ell_2) - c)^2 - 4\dot{\varphi}(\ell_2)} \right). \tag{3.4}$$

Proof. Assume by contradiction that

$$0 \le \lambda^- := \liminf_{u\to\ell_2^-} \frac{z(u)}{u-\ell_2} < \limsup_{u\to\ell_2^-} \frac{z(u)}{u-\ell_2} = \Lambda^- \le +\infty. \tag{3.5}$$

For a fixed $\gamma \in (\lambda^-, \Lambda^-)$, let $\{u_n\}_n \subset (0, \ell_2)$ be a sequence converging to ℓ_2 and such that

$$\frac{z(u_n)}{u_n - \ell_2} = \gamma, \qquad \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{z(u)}{u-\ell_2}\right)_{u=u_n} \le 0. \tag{3.6}$$

Since

$$\frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{z(u)}{u-\ell_2}\right)_{u=u_n} = \frac{1}{u_n - \ell_2}\left[\dot{z}(u_n) - \gamma\right], \tag{3.7}$$

we obtain $\dot{z}(u_n) \geq \gamma$. Replacing this inequality in (3.3) evaluated for $u = u_n$, and passing to the limit, we have

$$\gamma^2 - [h(\ell_2) - c]\gamma + \dot{\varphi}(\ell_2) \leq 0. \tag{3.8}$$

Similarly, taking a sequence $\{v_n\}_n \subset (0, \ell_2)$ converging to ℓ_2 and such that

$$\frac{z(v_n)}{v_n - \ell_2} = \gamma, \qquad \frac{\mathrm{d}}{\mathrm{d}u}\left(\frac{z(u)}{u - \ell_2}\right)_{u=v_n} \geq 0, \tag{3.9}$$

arguing as before, we obtain $\gamma^2 - [h(\ell_2) - c]\gamma + \dot{\varphi}(\ell_2) \geq 0$. Combining with (3.8) we get $\gamma^2 - [h(\ell_2) - c]\gamma + \dot{\varphi}(\ell_2) = 0$, which is impossible, due to the arbitrariness of $\gamma \in (\lambda^-, \Lambda^-)$. Then,

$$\lambda^- = \Lambda^- = \lim_{u \to \ell_2^-} \frac{z(u)}{u - \ell_2}. \tag{3.10}$$

Since (3.3) can be rewritten as follows:

$$\dot{z}(u) = h(u) - c - \frac{\varphi(u)}{u - \ell_2}\,\frac{u - \ell_2}{z(u)}, \tag{3.11}$$

then $\lambda^- = +\infty$ would imply that $\dot{z}(u) \to h(\ell_2) - c$ as $u \to \ell_2^-$, a contradiction. Hence, λ^- is real, nonnegative.

If $\lambda^- > 0$, then from (3.11) we get that $\dot{z}(u)$ admits limit as $u \to \ell_2^-$, and this limit must coincide with λ^-. Passing to the limit in (3.11) we get that λ^- is a root of

$$\lambda^2 + (c - h(\ell_2))\lambda + \dot{\varphi}(\ell_2) = 0. \tag{3.12}$$

If $\dot{\varphi}(\ell_2) = 0$, we immediately have $\dot{z}(u) \to \lambda^- = h(\ell_2) - c > 0$ as $u \to \ell_2^-$, since $\lambda^- > 0$. If $\dot{\varphi}(\ell_2) < 0$, since the above trinomial has two discordant zeros, we deduce that

$$\lambda^- = \frac{1}{2}\left(h(\ell_2) - c + \sqrt{(h(\ell_2) - c)^2 - 4\dot{\varphi}(\ell_2)}\right). \tag{3.13}$$

Summarizing, if $\lambda^- > 0$, then (3.13) holds.

Assume now that $\lambda^- = 0$. If $\dot{\varphi}(\ell_2) < 0$; from (3.11) we get $\dot{z}(u) \to +\infty$ as $u \to \ell_2^-$, a contradiction. Hence, $\dot{z}(\ell_2) = 0$ implies $\dot{\varphi}(\ell_2) = 0$. Moreover, observe that

$$h(\ell_2) - c = \lim_{u \to \ell_2^-} h(u) - c \leq \liminf_{u \to \ell_2^-}\left(h(u) - c - \frac{\varphi(u)}{z(u)}\right) = \liminf_{u \to \ell_2^-} \dot{z}(u) \leq \lambda^- = 0. \tag{3.14}$$

Thus, λ^- satisfies (3.13) also in the case $\lambda^- = 0$. □

By a simple change of variable ($v = \ell_1 + \ell_2 - u$ and $\tilde{z}(v) = -z(u)$), it is easy to check the validity of the following results in the case when φ is everywhere negative in (ℓ_1, ℓ_2).

Proposition 3.3. *Let $h, \varphi : [\ell_1, \ell_2] \to \mathbb{R}$ be continuous functions such that $\varphi(\ell_1) = \varphi(\ell_2) = 0$, $\varphi(u) < 0$ in (ℓ_1, ℓ_2), and $\dot{\varphi}(\ell_2)$ exists finite. Then, there exists a value c^*, satisfying the following estimate:*

$$2\sqrt{\dot{\varphi}(\ell_2)} + h(\ell_2) \le c^* \le 2\sqrt{\sup_{u\in[\ell_1,\ell_2)} \frac{\varphi(u)}{u-\ell_2}} + \max_{u\in[\ell_1,\ell_2]} h(u), \tag{3.15}$$

such that problem (3.3) *admits solutions if and only if $c \ge c^*$. Moreover, the solution is unique.*

Corollary 3.4. *Under the same assumptions of Proposition 3.3, if $\dot{\varphi}(\ell_1)$ exists finite, then also $\dot{z}(\ell_1)$ exists finite, and*

$$\dot{z}(\ell_1) = \frac{1}{2}\left(h(\ell_1) - c + \sqrt{(h(\ell_1) - c)^2 - 4\dot{\varphi}(\ell_1)}\right). \tag{3.16}$$

Combining the previous results, we are able to prove an existence theorem for problem (3.1).

Theorem 3.5. *Let $h, \varphi : [0,1] \to \mathbb{R}$ be continuous functions, with $\varphi(0) = \varphi(1) = 0$, φ differentiable at $u = \alpha$, satisfying condition* (2.1) *and such that*

$$\varphi(u)(\alpha - u) > 0, \quad \forall u \in (0,1) \setminus \{\alpha\}, \tag{3.17}$$

for a given constant $\alpha \in (0,1)$.

Then, there exists a value c^*, satisfying estimate (2.4)*, such that problem* (3.1) *admits solutions if and only if $c \ge c^*$. Moreover, the solution is unique.*

Proof. Consider the boundary value problem (3.3) for $\ell_1 = 0, \ell_2 = \alpha$. Since all the assumptions in Proposition 3.1 are satisfied, there exists a threshold value c_1^*, satisfying (3.2), such that the b.v.p (3.3), with $\ell_1 = 0$ and $\ell_2 = \alpha$, is uniquely solvable if and only if $c \ge c_1^*$. Similarly, considering the problem (3.3) for $\ell_1 = \alpha, \ell_2 = 1$, by Proposition 3.3, there is a threshold value c_2^*, satisfying (3.15), such that this problem is uniquely solvable if and only if $c \ge c_2^*$. Let $c^* := \max\{c_1^*, c_2^*\}$. Clearly, c^* satisfies (2.4). Moreover, if problem (3.3), with $\ell_1 = 0$ and $\ell_2 = 1$, is solvable for some c, then also problem (3.3) with $\ell_1 = 0, \ell_2 = \alpha$, and problem (3.3), with $\ell_1 = \alpha$ and $\ell_2 = 1$, are solvable for the same value c. Hence, $c \ge c^*$.

Conversely, let us fix $c \ge c^*$, and let $z \in C(0,1)$ be the function obtained gluing the unique solution of problem (3.3), with $\ell_1 = 0$ and $\ell_2 = \alpha$ and the unique solution of problem (3.3) with $\ell_1 = \alpha$ and $\ell_2 = 1$. In order to prove the assertion, it suffices to observe that, from Corollaries 3.2 and 3.4, the glued function z is differentiable at the point α. □

Remark 3.6. Notice that in the previous theorem the required differentiability of φ at α cannot be dropped. Indeed, in view of the proof, if the right derivative of φ at α does not coincide with the left one, then $\lambda^- \neq \lambda^+$ and z is not differentiable at α.

Concerning the equivalence between the solvability of problem (3.1) and the existence of t.w.s., the following result holds.

Theorem 3.7. *Let the assumptions* (1.2), (1.5), (1.6), *and* (2.1) *be satisfied. The existence of solutions to* (1.12) *for some c is equivalent to the solvability of problem* (3.1) *with the same c. If $h(u) \equiv 0$, then condition* (1.6) *can be removed.*

Proof. Let $u(\xi)$ be a t.w.s. of (1.1) with wave speed c, that is, a solution of problem (1.12) satisfying condition (1.3). Observe that if $u'(\xi_0) = 0$ with $0 < u(\xi_0) < 1$, for some $\xi_0 \in (a, b)$, then since $(D(u(\xi_0)u'(\xi_0)))' = -f(u(\xi_0)) < 0$, we have that ξ_0 is a point of proper local maximum for u when $u(\xi_0) \in (0, \alpha)$, while ξ_0 is a point of proper local minimum when $u(\xi_0) \in (\alpha, 1)$. Hence, assume now by contradiction the existence of a value $\xi^* \in (a, b)$ such that $\alpha < u(\xi^*) < 1$ and $u'(\xi^*) > 0$. By the boundary condition $u(b^-) = 0$, we get the existence of a value $\bar{\xi} > \xi^*$ such that $u(\bar{\xi}) = 1$ and $u'(\bar{\xi}) = 0$. Integrating (1.12) in $(a, \bar{\xi})$, we obtain

$$0 = D\left(u\left(\bar{\xi}\right)\right)u'\left(\bar{\xi}\right) - D(u(a))u'(a) = -\int_a^{\bar{\xi}} f(u(\xi))d\xi, \tag{3.18}$$

in contradiction with condition (1.2). Similarly, if $u'(\xi^*) > 0$ with $0 < u(\xi^*) < \alpha$ for some $\xi^* \in (a, b)$, then there exists a value $\bar{\xi} < \xi^*$ such that $u(\bar{\xi}) = 0$ and $u'(\bar{\xi}) = 0$. Integrating (1.12) in $(\bar{\xi}, b)$, we get again a contradiction. Therefore, $u'(\xi) < 0$ whenever $u(\xi) \in (0, 1)$, $u(\xi) \neq \alpha$.

Put $\xi_1 := \inf\{\xi : u(\xi) = \alpha\}$ and $\xi_2 := \sup\{\xi : u(\xi) = \alpha\}$. If $\xi_1 < \xi_2$ then u is constant in (ξ_1, ξ_2), and we can define $\tilde{u}(\xi) := u(\xi)$ if $\xi \leq \xi_1$; $\tilde{u}(\xi) := u(\xi + \xi_2 - \xi_1)$ if $\xi > \xi_1$. Of course, since (1.12) is autonomous, $\tilde{u}$ is a t.w.s. of (1.1) in its existence interval $\tilde{I} := (a, b - \xi_2 + \xi_1)$ and satisfies $\tilde{u}'(\xi) < 0$, for every $\xi \in \tilde{I}$, $\xi \neq \xi_1$. Hence, $\tilde{u}$ is strictly decreasing and invertible in $\tilde{I}$. Let $z(u) := D(u)\tilde{u}'(\xi(u))$, $u \in (0, 1)$. It is easy to see that the function $z(u)$ is a solution of (3.1) for the same value of c.

Now assume that $0 < \alpha < 1$, and let $z(u)$ be a solution of (3.1) for some admissible real value c. Let $u_1(\xi)$ be the unique solution of the Cauchy problem:

$$\begin{gathered} u' = \frac{z(u)}{D(u)}, \quad 0 < u < \alpha, \\ u(0) = \frac{\alpha}{2}, \end{gathered} \tag{3.19}$$

defined on its maximal existence interval (t_1, t_2). Since $u'(\xi) < 0$ whenever $\xi \in (t_1, t_2)$, we have that $u(t_2^-) = 0$ and $u(t_1^+) = \alpha$. Moreover, according to (1.6) and Corollary 3.2, the limit $\lim_{\xi \to t_1^+} u'(\xi)$ exists and it is not zero, implying that $t_1 \in \mathbb{R}$. Consider now the unique solution $u_2(\xi)$ of the initial value problem:

$$\begin{gathered} u' = \frac{z(u)}{D(u)}, \quad \alpha < u < 1, \\ u(0) = \frac{1 + \alpha}{2}, \end{gathered} \tag{3.20}$$

on its maximal existence interval (τ_2, τ_1). Again, $u_2'(\xi) < 0$ for $\xi \in (\tau_1, \tau_2)$ implies $u(\tau_1^+) = 1$ and $u(\tau_2^-) = \alpha$. Moreover, according to (1.6) and Corollary 3.4, we obtain that $\lim_{\xi \to \tau_2^-} u_2'(\xi) = \lim_{\xi \to t_1^+} u_1(\xi)$; hence, also the former limit is not zero and $\tau_2 \in \mathbb{R}$. Then, as we made above, with a suitable shift, it is possible to glue the two functions $u_i(\xi)$, $i = 1, 2$, in such a way to have a unique $u = u(\xi)$, defined on some interval $(a, b) \subseteq \mathbb{R}$, with $u \in C^1((a, b))$, $u(\xi) \in (0, 1)$ on the whole (a, b), and u satisfying (1.12) and (1.3).

When $\alpha = 0$ or $\alpha = 1$ one considers just one of the Cauchy problems above defined, whose solution is a t.w.s. for (1.1).

Finally, assume $h \equiv 0$. It is well known (see, e.g., [2]) that c^* is always strictly positive in this case. Moreover, (see, e.g., [2, Lemma 1]),

$$\lim_{\xi \to t_1^+} u'(\xi) = \lim_{\xi \to \tau_2^-} u'(\xi) = \frac{2f(\alpha)}{c + \sqrt{c^2 - 4f(\alpha)\dot{D}(\alpha)}}. \tag{3.21}$$

Hence, both limits are finite and nonzero independently on the value of $\dot{D}(\alpha)$. So condition (1.6) can be removed. □

Remark 3.8. If $D(0) > 0$, then $b = +\infty$. Indeed, $\dot{z}(u) > h(u) - c$, for $u \in (0, \alpha)$, implies $z(u) > \int_0^u [h(s) - c]\mathrm{d}s$. Therefore, if $u(t_0) = \alpha/2$ for some $t_0 \in \mathbb{R}$, we have that

$$b - t_0 = \int_{\alpha/2}^{0} \dot{t}(u)\mathrm{d}u = \int_0^{\alpha/2} \frac{D(u)}{-z(u)}\mathrm{d}u \geq \int_0^{\alpha/2} \frac{D(u)}{\int_0^u [c - h(s)]\mathrm{d}s}\mathrm{d}u. \tag{3.22}$$

Being an admissible value, the wave speed c satisfies condition (2.4); hence, there exists $\sigma > 0$ such that $\int_0^u [h(s) - c]\mathrm{d}s \geq \sigma u$ for $u \in (0, \alpha)$, implying $b = +\infty$. With a similar argument, it can be showed that $D(1) < 0$ implies $a = -\infty$.

In light of the above equivalence result, Theorem 2.1 is an immediate consequence of Propositions 3.1, 3.3 and Theorem 3.5.

4. The Continuous Dependence Results

As we mentioned in the Introduction section, both the dependence on the parameter k of the minimal wave speed $c^*(k)$ and of the wave profile $U(\xi; c, k)$ corresponding to the speed c were already studied in [11] for the diffusive case, that is, in case $D(u; k) \geq 0$, for every $u \in (0, 1)$. More in detail, the following result holds (see [11, Theorems 4.1 and 4.2]).

Theorem 4.1. *Let $h, D, f : [0, 1] \times [0, 1] \to \mathbb{R}$ be continuous functions, with $D(\cdot; k) \in C^1[0, 1]$ for every $k \in [0, 1]$ and $D(u; k) > 0$ for every $(u, k) \in (0, 1) \times [0, 1]$. Assume that* (1.8) *and* (2.1) *are satisfied for all $k \in [0, 1]$. Then, the threshold value c^* is a l.s.c. function in $[0, 1]$. Moreover, if for some $k_0 \in [0, 1]$*

$$\limsup_{(u,k) \to (0,k_0)} \frac{\varphi(u; k) - \varphi(u; k_0)}{u} \leq 0, \tag{4.1}$$

then c^* is continuous at $k = k_0$.

By means of a change of variable, in the aggregative case, that is, in case $D(u; k) < 0$ for every $u \in (0, 1)$, it can be proved that the analogous result holds.

Theorem 4.2. *Let $h, D, f : [0,1] \times [0,1] \to \mathbb{R}$ be continuous functions, with $D(\cdot;k) \in C^1[0,1]$ for every $k \in [0,1]$ and $D(u;k) < 0$ for every $(u,k) \in (0,1) \times [0,1]$. Assume that* (1.8) *and* (2.1) *are satisfied for all $k \in [0,1]$. Then, the threshold value c^* is a l.s.c. function in $[0,1]$. Moreover, if for some $k_0 \in [0,1]$*

$$\liminf_{(u,k)\to(1,k_0)} \frac{\varphi(u;k) - \varphi(u;k_0)}{1-u} \geq 0, \tag{4.2}$$

then c^* is continuous at $k = k_0$.

Combining the results of Section 3 with Theorems 4.1 and 4.2, we are able to prove Theorem 2.2.

Proof of Theorem 2.2. In view of the equivalence proved in Theorem 3.7, we analyze the continuous dependence of the threshold value $c^*(k)$ for the problem:

$$\begin{gathered} \dot{z}(u) = h(u;k) - c - \frac{\varphi(u;k)}{z(u)}, \quad \text{for every } u \in (0,1) \setminus \{\alpha(k)\}, \\ z(0^+) = z(1^-) = 0, \\ z(u)\varphi(u) < 0, \quad \forall u \in (0,1),\ u \neq \alpha(k), \end{gathered} \tag{4.3}$$

where c is an unknown real constant.

For every $k \in [0,1)$ we have $0 < \alpha(k) \leq 1$; hence, we can consider the boundary value problem:

$$\begin{gathered} \dot{z}(u) = h(u;k) - c - \frac{\varphi(u;k)}{z(u)}, \quad u \in (0,\alpha(k)), \\ z(0^+) = z(\alpha(k)^-) = 0, \\ z(u) < 0, \quad \forall u \in (0,\alpha(k)). \end{gathered} \tag{$P_{1,k}$}$$

Put $\widetilde{h}(u;k) := \alpha(k)h(\alpha(k)u;k)$ and $\widetilde{\varphi}(u;k) := \alpha(k)\varphi(\alpha(k)u;k)$, problem (3.22) is equivalent to the normalized one:

$$\begin{gathered} \dot{w}(u) = \widetilde{h}(u;k) - c\alpha(k) - \frac{\widetilde{\varphi}(u;k)}{w(u)}, \quad u \in (0,1), \\ w(0^+) = w(1^-) = 0, \\ w(u) < 0, \quad \forall u \in (0,1). \end{gathered} \tag{$\widetilde{P}_{1,k}$}$$

Indeed, a function $z(u)$ is a solution of (3.22) if and only if the function $w(u) := z(\alpha(k)u)$ is a solution of $(P_{1,k})$. Hence, if $c_1^*(k)$ and $\widetilde{c}_1^*(k)$ denote, respectively, the minimal values of the parameter c for which (3.22) and $(P_{1,k})$ are solvable, we have

$$c_1^*(k) = \frac{1}{\alpha(k)}\widetilde{c}_1^*(k). \tag{4.4}$$

Since $\alpha(k)$ is continuous and positive in $[0,1)$, from Theorem 4.1 we have that $c_1^*(k)$ is lower semicontinuous at k_0 for every $k_0 \in [0,1)$.

Similarly, for every $k \in (0,1]$, consider the problem:

$$\begin{gathered}\dot{z}(u) = h(u;k) - c - \frac{\varphi(u;k)}{z(u)}, \quad u \in (\alpha(k),1),\\ z\left(\alpha(k)^+\right) = z\left(1^-\right) = 0,\\ z(u) > 0, \quad \forall u \in (\alpha(k),1).\end{gathered} \tag{$P_{2,k}$}$$

Using an argument analogous to the previous one and taking into account that, in this case, $0 \le \alpha(k) < 1$, we deduce that the threshold value $c_2^*(k)$ for problem $(\widetilde{P}_{1,k})$ is a lower semicontinuous function at every $k_0 \in (0,1]$.

Let $k \in (0,1)$ and let c^* be the threshold value for problem (4.3). Since $c^*(k) = \max\{c_1^*(k), c_2^*(k)\}$ (see the proof of Theorem 3.5), with c_1^*, c_2^* lower semicontinuous functions, then $c^*(k)$ is lower semicontinuous at every $k_0 \in (0,1)$.

If $k_0 = 0$, then it results in $c^*(0) = c_1^*(0)$, and

$$\liminf_{k\to 0} c^*(k) = \liminf_{k\to 0}\left(\max\left\{c_1^*(k), c_2^*(k)\right\}\right) \ge \liminf_{k\to 0} c_1^*(k) \ge c_1^*(0) = c^*(0). \tag{4.5}$$

The case $k_0 = 1$ can be treated in a similar way, and the semicontinuity of c^* in $[0,1]$ is proved.

Concerning the study of the continuity of c^* under condition (2.5), let us first consider a value $k_0 \in (0,1)$. Analogously to the arguments above developed, we can consider the problem (3.22) and the normalized one $(P_{1,k})$. Notice that, if φ satisfies condition (4.1), then also the function $\widetilde{\varphi}$ does. Indeed, we have

$$\begin{aligned}\limsup_{(u,k)\to(0,k_0)} \frac{\widetilde{\varphi}(u;k) - \widetilde{\varphi}(u;k_0)}{u} &= \limsup_{(u,k)\to(0,k_0)} \frac{\alpha(k)\varphi(\alpha(k)u;k) - \alpha(k_0)\varphi(\alpha(k_0)u;k_0)}{u}\\ &= \limsup_{(v,k)\to(0,k_0)} \alpha^2(k)\frac{\varphi(v;k) - \varphi(v;k_0)}{v}\\ &\quad + \lim_{(u,k)\to(0,k_0)}\left(\alpha^2(k)\frac{\varphi(\alpha(k)u;k_0)}{u\alpha(k)} - \alpha^2(k_0)\frac{\varphi(\alpha(k_0)\,u;k_0}{u\alpha(k_0)}\right) \le 0,\end{aligned} \tag{4.6}$$

by the positivity and the continuity of $\alpha(k)$ and the differentiability of $\varphi(u;k_0)$ at 0. Hence, Theorem 4.1 assures that the function $\widetilde{c}_1^*(k)$ is continuous at k_0, and in force of (4.4) we conclude that $c_1^*(k)$ is continuous too. Analogously, considering the problem $(\widetilde{P}_{1,k})$, we can prove that the threshold value $c_2^*(k)$ is continuous at every $k_0 \in (0,1)$. Since $c^*(k) = \max\{c_1^*(k), c_2^*(k)\}$ (see the proof of Theorem 3.5), then $c^*(k)$ is continuous at every $k_0 \in (0,1)$.

It remains to study the continuity of c^* at the values $k_0 = 0$ and $k_0 = 1$, where the dynamic change its nature. We limit ourselves in considering the value $k_0 = 0$, since the proof for $k_0 = 1$ is analogous.

Let $\epsilon > 0$ be fixed, and observe that, by assumption (2.6), real values $\delta = \delta(\epsilon) > 0$ and $\overline{k} = \overline{k}(\epsilon) \in (0,1)$ exist, such that

$$\frac{\varphi(u;k)}{u-1} \leq \dot{\varphi}(1;k) + \epsilon, \quad \text{for every } u \in (1-\delta, 1),\ k \in \left(0, \overline{k}\right). \tag{4.7}$$

Let $\widetilde{k} = \widetilde{k}(\epsilon) \in (0, \overline{k})$ be such that $\alpha(k) \in (1-\delta, 1)$ for every $k \in (0, \widetilde{k})$. Then

$$\sup_{u \in [\alpha(k), 1)} \frac{\varphi(u;k)}{u-1} \leq \dot{\varphi}(1;k) + \epsilon, \quad \text{for every } k \in \left(0, \widetilde{k}\right). \tag{4.8}$$

Taking estimate (3.15) into account, with $\ell_1 = \alpha(k)$ and $\ell_2 = 1$, we get

$$c_2^*(k) \leq 2\sqrt{\dot{\varphi}(1;k) + \epsilon} + \max_{u \in [\alpha(k), 1]} h(u;k), \quad \text{for every } k \in \left(0, \widetilde{k}\right), \tag{4.9}$$

implying

$$\limsup_{k \to 0} c_2^*(k) \leq 2\sqrt{\limsup_{k \to 0} \dot{\varphi}(1;k) + \epsilon} + h(1;0). \tag{4.10}$$

By the arbitrariness of $\epsilon > 0$, assumption (2.7), and estimate (3.2) with $\ell_1 = 0$ and $\ell_2 = \alpha(k)$, we have

$$\limsup_{k \to 0} c_2^*(k) \leq 2\sqrt{\dot{\varphi}(0;0)} + h(0;0) \leq c_1^*(0) = c^*(0). \tag{4.11}$$

Therefore, since $c_1^*(k) \to c_1^*(0) = c^*(0)$ as $k \to 0$, we conclude that

$$\limsup_{k \to 0} c^*(k) = \limsup_{k \to 0} \max\left\{c_1^*(k), c_2^*(k)\right\} = c^*(0), \tag{4.12}$$

and the function c^* is upper semicontinuous in $k_0 = 0$. Taking into account the lower semicontinuity of c^* in the whole interval $[0,1]$, it follows the continuity in $k_0 = 0$. The proof for the value $k_0 = 1$ is similar. □

In order to prove Theorem 2.6 about the convergence of the profiles, we need some preliminary results.

Let $z(u;c,k)$ denote the unique solution of problem (4.3), emphasizing the dependence on the parameters c and k. The following result holds.

Lemma 4.3. *Let* $\alpha : [0,1] \to [0,1]$ *and* $h, D, f : [0,1] \times [0,1] \to \mathbb{R}$ *be continuous functions satisfying conditions* (1.8), (1.9), (1.10), *and* (2.1) *for every* $k \in [0,1]$*. Let* $\gamma : [0,1] \to [0,1]$ *be a continuous function such that* $\gamma(k) \geq c^*(k)$ *for every* $k \in [0,1]$*. Finally, let* $z(u;\gamma(k),k)$ *be the unique solution of problem* (4.3)*.*

Then, for every $k \in (0,1)$*, there exists a real value* $\lambda(k)$ *such that the function*

$$\psi(u;k) := \begin{cases} \dfrac{z(u;\gamma(k),k)}{D(u;k)}, & \text{for } u \in (0,1),\ u \neq \alpha(k), \\ \lambda(k), & \text{for } u = \alpha(k) \end{cases} \tag{4.13}$$

is continuous in $(0,1) \times (0,1)$*.*

Proof. Let $(u_0, k_0) \in (0,1) \times (0,1)$ be fixed. We divide the proof in two cases.

Case 1 ($u_0 \neq \alpha(k_0)$). Assume $u_0 < \alpha(k_0)$ (the proof in the other case is analogous), and let $[a,b] \subset (0, \alpha(k_0))$ be a compact interval containing the value u_0. Put $z_k(u) := z(u;\gamma(k),k)$ and let $w_k(u) := w(u;\gamma(k)\alpha(k),k)$ denote the solution of problem $(P_{1,k})$ (see the proof of Theorem 2.2). By applying [11, Theorem 5.1] to the problem $(P_{1,k})$ in $[a, 2b/(b+\alpha(k_0))] \subset (0,1)$, we get the convergence of $w_k(u)$ to $w_{k_0}(u)$ as $k \to k_0$, uniformly in $[a, 2b/(b+\alpha(k_0))]$; that is, for a fixed $\epsilon > 0$,

$$|w_k(u) - w_{k_0}(u)| < \frac{1}{2}\epsilon \quad \text{for every } u \in \left[a, \frac{2b}{b+\alpha(k_0)}\right]. \tag{4.14}$$

Taking account of the continuity of α and of the uniform continuity of w_{k_0}, there exists a value $\eta > 0$ such that for every $k \in (k_0 - \eta, k_0 + \eta)$ we have $\alpha(k) > (1/2)(b + \alpha(k_0))$ and

$$\left| w_{k_0}\left(\frac{u}{\alpha(k)}\right) - w_{k_0}\left(\frac{u}{\alpha(k_0)}\right) \right| < \frac{1}{2}\epsilon, \quad \text{for every } u \in [a,b]. \tag{4.15}$$

Since $u/\alpha(k) \in [a, 2b/(b+\alpha(k_0))]$ whenever $u \in [a,b]$ and $k \in (k_0 - \eta, k_0 + \eta)$, we deduce that

$$|z_k(u) - z_{k_0}(u)| \leq \left| w_k\left(\frac{u}{\alpha(k)}\right) - w_{k_0}\left(\frac{u}{\alpha(k)}\right) \right| + \left| w_{k_0}\left(\frac{u}{\alpha(k)}\right) - w_{k_0}\left(\frac{u}{\alpha(k_0)}\right) \right| \leq \epsilon, \tag{4.16}$$

for every $k \in (k_0 - \eta, k_0 + \eta)$ and $u \in [a,b]$, that is, the uniform convergence of z_k to z_{k_0} in $[a,b]$. Taking account of the uniform continuity of z_{k_0} in $[0,1]$, we get $z_k(u) \to z_{k_0}(u_0)$ as $(u,k) \to (u_0,k_0)$. Finally, since $D(u_0,k_0) \neq 0$, by the continuity of $D(u;k)$ we also get

$$\lim_{(u,k) \to (u_0,k_0)} \frac{z_k(u)}{D(u;k)} = \frac{z_{k_0}(u_0)}{D(u_0;k_0)}, \tag{4.17}$$

and the continuity of the function ψ at (u_0,k_0) is proved.

Notice that, for the opposite case $u_0 > \alpha(k_0)$, the usual change of variable $v := 1 - u$ allows to apply [11, Theorem 5.1].

Case 2 ($u_0 = \alpha(k_0)$). For every $k \in (0,1)$, let

$$\lambda(k) = -\frac{2f(\alpha(k);k)}{-h(\alpha(k);k) + \gamma(k) + \sqrt{\left[h(\alpha(k);k) - \gamma(k)\right]^2 - 4f(\alpha(k);k)\dot{D}(\alpha(k);k)}}. \tag{4.18}$$

Notice that the function λ is well defined since by assumption (1.6) we have $\dot{D}(\alpha(k);k) < 0$. Moreover, λ is continuous and negative for every $k \in (0,1)$. Since

$$\lambda(k) = \frac{h(\alpha(k);k) - \gamma(k) + \sqrt{\left[h(\alpha(k);k) - \gamma(k)\right]^2 - 4f(\alpha(k);k)\dot{D}(\alpha(k);k)}}{2\dot{D}(\alpha(k);k)}, \tag{4.19}$$

from Corollary 3.2 it follows that

$$\dot{z}_k(\alpha(k)) = \lambda(k)\dot{D}(\alpha(k);k). \tag{4.20}$$

Let us fix a value $\epsilon > 0$ and define

$$\begin{aligned} \xi_\epsilon(u;k) &= (\lambda(k_0) - \epsilon)D(u;k), \\ \Phi_\epsilon(u;k) &= \dot{\xi}_\epsilon(u;k) - h(u;k) + \gamma(k) + \frac{f(u;k)D(u;k)}{\xi_\epsilon(u;k)}. \end{aligned} \tag{4.21}$$

We have

$$\begin{aligned} \Phi_\epsilon(u;k) &= \lambda(k_0)\dot{D}(u;k) - h(u;k) + \gamma(k) + \frac{f(u;k)}{\lambda(k_0) - \epsilon} - \epsilon\dot{D}(u;k) \\ &= \lambda(k_0)\dot{D}(u;k) - h(u;k) + \gamma(k) + \frac{f(u;k)}{\lambda(k_0)} + \frac{\epsilon f(u;k)}{\lambda(k_0)(\lambda(k_0) - \epsilon)} - \epsilon\dot{D}(u;k). \end{aligned} \tag{4.22}$$

From the definition of $\lambda(k)$, we get

$$\lambda(k)\dot{D}(\alpha(k);k) - h(\alpha(k);k) + \gamma(k) + \frac{f(\alpha(k);k)}{\lambda(k)} = 0. \tag{4.23}$$

Therefore,

$$\lim_{(u,k)\to(\alpha(k_0),k_0)} \Phi_\epsilon(u;k) = \epsilon\frac{f(\alpha(k_0);k_0)}{\lambda(k_0)(\lambda(k_0) - \epsilon)} - \epsilon\dot{D}(\alpha(k_0);k_0) =: 2\sigma > 0. \tag{4.24}$$

Hence, there exists a value $\delta_1 > 0$ such that

$$\Phi_\epsilon(u;k) \geq \sigma > 0, \quad \text{whenever } |u - \alpha(k_0)| < \delta_1, \quad |k - k_0| < \delta_1, \tag{4.25}$$

that is,

$$\dot{\xi}_\epsilon(u;k) > h(u;k) - \gamma(k) - \frac{f(u;k)D(u;k)}{\xi_\epsilon(u;k)}, \quad \text{whenever } |u-\alpha(k_0)| < \delta_1,\ |k-k_0| < \delta_1. \tag{4.26}$$

Take now $\delta \in (0,\delta_1]$ such that $|\alpha(k)-\alpha(k_0)| < (1/2)\delta_1$ and $\lambda(k) > \lambda(k_0)-\epsilon$ whenever $|k-k_0| < \delta$. Observe that, by (4.20), we have $\dot{z}_k(\alpha(k)) < \dot{\xi}_\epsilon(\alpha(k);k)$. Since $z_k(\alpha(k)) = \xi_\epsilon(\alpha(k);k) = 0$, then for every $k \in (k_0-\delta, k_0+\delta)$, there exists a value $\rho = \rho(k) \in (0,\alpha(k))$ such that

$$\begin{aligned} 0 > z_k(u) > \xi_\epsilon(u;k), &\quad \text{for every } u \in \big(\alpha(k)-\rho(k),\alpha(k)\big), \\ 0 < z_k(u) < \xi_\epsilon(u;k), &\quad \text{for every } u \in \big(\alpha(k),\alpha(k)+\rho(k)\big). \end{aligned} \tag{4.27}$$

Assume, by contradiction, the existence of a value $k \in (k_0-\delta,k_0+\delta)$ and of a point $v(k) \in [\alpha(k_0)-\delta_1,\alpha(k))$ such that

$$z_k(v(k)) = \xi_\epsilon(v(k);k), \qquad z_k(u) > \xi_\epsilon(u;k) \quad \text{in } (v(k),\alpha(k)). \tag{4.28}$$

Then,

$$\begin{aligned} \dot{z}_k(v(k)) &= h(v(k);k) - \gamma(k) - \frac{f(v(k);k)D(v(k);k)}{z_k(v(k))} \\ &= h(v(k);k) - \gamma(k) - \frac{f(v(k);k)D(v(k);k)}{\xi_\epsilon(v(k);k)} < \dot{\xi}_\epsilon(v(k);k), \end{aligned} \tag{4.29}$$

in contradiction with the definition of $v(k)$. Therefore, we deduce that

$$\xi_\epsilon(u;k) < z_k(u) < 0, \quad \text{for every } u \in [\alpha(k_0)-\delta_1,\alpha(k)),\ k \in [k_0-\delta,k_0+\delta]. \tag{4.30}$$

Similarly, we can prove that

$$\xi_\epsilon(u;k) > z_k(u) > 0, \quad \text{for every } u \in (\alpha(k),\alpha(k_0)+\delta_1], \quad k \in [k_0-\delta,k_0+\delta]. \tag{4.31}$$

The previous conditions (4.30) and (4.31) imply that

$$\frac{z_k(u)}{D(u;k)} > \frac{\xi_\epsilon(u;k)}{D(u;k)} = \lambda(k_0)-\epsilon, \quad \text{for every } u \in [\alpha(k_0)-\delta_1,\alpha(k_0)+\delta_1] \setminus \{\alpha(k)\}. \tag{4.32}$$

Let us now take $\epsilon < -\lambda(k_0)$ and consider the functions

$$\begin{aligned} \eta_\epsilon(u;k) &:= (\lambda(k_0)+\epsilon)D(u;k), \\ \Psi_\epsilon(u;k) &:= \dot{\eta}_\epsilon(u;k) - h(u;k) + \gamma(k) + \frac{f(u;k)D(u;k)}{\eta_\epsilon(u;k)}. \end{aligned} \tag{4.33}$$

We have

$$\Psi_\epsilon(u;k) = \lambda(k_0)\dot{D}(u;k) - h(u;k) + \gamma(k) + \frac{f(u;k)}{\lambda(k_0)+\epsilon} + \epsilon\dot{D}(u;k)$$
$$= \lambda(k_0)\dot{D}(u;k) - h(u;k) + \gamma(k) + \frac{f(u;k)}{\lambda(k_0)} - \frac{\epsilon f(u;k)}{\lambda(k_0)(\lambda(k_0)+\epsilon)} + \epsilon\dot{D}(u;k), \tag{4.34}$$

and, from (4.23), we get

$$\lim_{(u,k)\to(\alpha(k_0),k_0)} \Psi_\epsilon(u;k) = -\epsilon\frac{f(\alpha(k_0);k_0)}{\lambda(k_0)(\lambda(k_0)+\epsilon)} + \epsilon\dot{D}(\alpha(k_0);k_0) =: -2\widetilde{\sigma} < 0. \tag{4.35}$$

Hence, there exists a value $\widetilde{\delta}_1 > 0$ such that

$$\Psi_\epsilon(u;k) \le -\sigma < 0, \quad \text{whenever } |u-\alpha(k_0)| < \widetilde{\delta}_1,\ |k-k_0| < \widetilde{\delta}_1, \tag{4.36}$$

that is,

$$\dot{\eta}_\epsilon(u;k) < h(u;k) - \gamma(k) - \frac{f(u;k)D(u;k)}{\eta_\epsilon(u;k)}, \quad \text{whenever } |u-\alpha(k_0)| < \widetilde{\delta}_1,\ |k-k_0| < \widetilde{\delta}_1. \tag{4.37}$$

Let $\widetilde{\delta} \in (0,\widetilde{\delta}_1]$ satisfying $|\alpha(k)-\alpha(k_0)| < (1/2)\widetilde{\delta}_1$ and $\lambda(k) < \lambda(k_0)+\epsilon$ for $|k-k_0| < \widetilde{\delta}$. Observe that, by (4.20), we have $\dot{z}_k(\alpha(k)) > \dot{\eta}_\epsilon(\alpha(k);k)$. Then, since $z_k(\alpha(k)) = \eta_\epsilon(\alpha(k);k) = 0$, for every $k \in (k_0-\widetilde{\delta}, k_0+\widetilde{\delta})$, there exists a value $\widetilde{\rho} = \widetilde{\rho}(k) \in (0,\alpha(k))$, such that

$$z_k(u) < \eta_\epsilon(u;k), \quad \text{for every } u \in \left(\alpha(k)-\widetilde{\rho}(k), \alpha(k)\right),$$
$$z_k(u) > \eta_\epsilon(u;k), \quad \text{for every } u \in \left(\alpha(k), \alpha(k)+\widetilde{\rho}(k)\right). \tag{4.38}$$

Assume, by contradiction, the existence of $k \in (k_0-\widetilde{\delta}, k_0+\widetilde{\delta})$ and of $\widetilde{v}(k) \in [\alpha(k_0)-\widetilde{\delta}_1, \alpha(k))$ such that

$$z_k(\widetilde{v}(k)) = \eta_\epsilon(\widetilde{v}(k);k), \quad z_k(u) < \eta_\epsilon(u;k) \quad \text{in } (\widetilde{v}(k),\alpha(k)). \tag{4.39}$$

Then,

$$\dot{z}_k(\widetilde{v}(k)) = h(\widetilde{v}(k);k) - \gamma(k) - \frac{f(\widetilde{v}(k);k)D(\widetilde{v}(k);k)}{z_k(\widetilde{v}(k))}$$
$$= h(\widetilde{v}(k);k) - \gamma(k) - \frac{f(\widetilde{v}(k);k)D(\widetilde{v}(k);k)}{\eta_\epsilon(\widetilde{v}(k);k)} > \dot{\eta}_\epsilon(\widetilde{v}(k);k), \tag{4.40}$$

in contradiction with the definition of $\widetilde{v}(k)$. Therefore, we deduce that

$$z_k(u) < \eta_\epsilon(u;k) < 0, \quad \text{for every } u \in \left[\alpha(k_0) - \widetilde{\delta}_1, \alpha(k)\right), \ k \in \left[k_0 - \widetilde{\delta}, k_0 + \widetilde{\delta}\right]. \tag{4.41}$$

Similarly one can prove that

$$z_k(u) > \eta_\epsilon(u;k) > 0, \quad \text{for every } u \in \left(\alpha(k), \alpha(k_0) + \widetilde{\delta}_1\right], \ k \in \left[k_0 - \widetilde{\delta}, k_0 + \widetilde{\delta}\right]. \tag{4.42}$$

The previous conditions (4.41) and (4.42) imply that

$$\frac{z_k(u)}{D(u;k)} < \frac{\eta_\epsilon(u;k)}{D(u;k)} = \lambda(k_0) + \epsilon, \quad \text{for every } u \in \left[\alpha(k_0) - \widetilde{\delta}_1, \alpha(k_0) + \widetilde{\delta}_1\right] \setminus \{\alpha(k)\}. \tag{4.43}$$

Taking the definition of function φ into account, since the function $\lambda(k)$ is continuous, by (4.32), (4.43), and the arbitrariness of $\epsilon > 0$, we deduce that

$$\lim_{(u,k) \to (\alpha(k_0), k_0)} \varphi(u;k) = \lambda(k_0) = \varphi(\alpha(k_0); k_0), \tag{4.44}$$

and this concludes the proof. □

Remark 4.4. The same reasoning developed in Case 1 of the proof of Lemma 4.3 also allows to discuss the continuity of φ in all points $(u, 0)$ or $(u, 1)$ with $u \in (0,1)$. More precisely, φ is continuous in every set $[a,b] \times [\alpha, \beta]$ where $[a,b] \subset (0,1)$, $[\alpha,\beta] \subset [0,1]$ and $\alpha(k) \notin [a,b]$ for all $k \in [\alpha, \beta]$.

Lemma 4.5 (see [13, Lemma 2.5]). *Let $(g_n)_{n \geq 0}$, $g_n : \mathbb{R} \to [0,1]$, be a sequence of continuous decreasing functions satisfying*

$$\lim_{t \to -\infty} g_n(t) = 1, \qquad \lim_{t \to +\infty} g_n(t) = 0, \quad n \geq 0. \tag{4.45}$$

Assume that $g_n(t) \to g_0(t)$ for every t in a dense subset of the interval $(\alpha_0, \beta_0) := \{t \in \mathbb{R} : 0 < g_0(t) < 1\}$. Then $g_n \to g_0$ uniformly on $\mathbb{R}$.

Proof of Theorem 2.6. Let $k_0 \in [0,1]$. According to Theorem 2.2, $\gamma(k_0) \geq c^*(k_0)$, and then the profile $u_{k_0}(\xi)$ is well defined.

For every $k \in [0,1]$, let $z_k(u) = z(u; \gamma(k), k)$ be the unique solution of problem (4.3), and let (a_k, b_k) be the maximal existence interval of the profile $u_k(\xi)$, with $-\infty \leq a_k < 0 < b_k \leq +\infty$, that is, $u_k(a_k^+) = 1$, $u_k(b^-) = 0$. We firstly prove the pointwise convergence in every compact interval contained in (a_{k_0}, b_{k_0}). To this aim, let $[\xi_0, \xi_1] \subset (a_{k_0}, b_{k_0})$, with $\xi_0 < 0 < \xi_1$, and let $u_{k_0}([\xi_0, \xi_1]) =: [v_1, v_0] \subset (0,1)$. First we consider the case when $k_0 \in (0,1)$ and fix $\delta > 0$

such that $[k_0 - \delta, k_0 + \delta] \subset (0,1)$. For $k \in [k_0 - \delta, k_0 + \delta]$, $u_k(\xi)$ is the unique solution of the initial value problem (see Theorem 3.7):

$$u'(\xi) = \varphi(u(\xi); k),$$
$$u(0) = u_0, \tag{4.46}$$

where $\varphi(u; k)$ is the function defined in Lemma 4.3 and φ is continuous for $(u, k) \in (0,1) \times (0,1)$. Further, u_k is a decreasing C^1-function on (a_k, b_k). For every k put

$$\tau_k := \max\left\{\xi_0, u_k^{-1}(v_0)\right\} < 0, \qquad t_k := \min\left\{\xi_1, u_k^{-1}(v_1)\right\} > 0. \tag{4.47}$$

Let us consider the truncated functions $\widetilde{u}_k(\xi) := \max\{v_1, \min\{u_k(\xi), v_0\}\}$, defined in $[\xi_0, \xi_1]$. Of course, if $\tau_k > \xi_0$ $[t_k < \xi_1]$ then $\widetilde{u}_k$ is constant in (ξ_0, τ_k) $[(t_k, \xi_1)]$.

Since $\varphi : [v_0, v_1] \times [k_0 - \delta, k_0 + \delta] \to \mathbb{R}$ is bounded, then the functions $\widetilde{u}_k(\xi)$ are equicontinuous in $[\xi_0, \xi_1]$, for every $k \in [k_0 - \delta, k_0 + \delta]$. □

Hence, for a fixed sequence $(k_n)_n$ converging to k_0, there exists a subsequence, denoted again $(k_n)_n$, such that $(\widetilde{u}_{k_n})_n$ is uniformly convergent. Possibly passing to further subsequence, we can also assume that $(\tau_{k_n})_n$ and $(t_{k_n})_n$, respectively, converge to some $\widetilde{\tau}, \widetilde{t} \in [\xi_0, \xi_1]$.

Notice that $\widetilde{\tau} < 0 < \widetilde{t}$. Indeed, whenever $\widetilde{\tau}_k > \xi_0$, we have $\widetilde{u}_{k_n}(\tau_{k_n}) = v_0$, and, since $\widetilde{u}_{k_n}(0) = u_0$ for every n, we get $\widetilde{\tau} < 0$, owing to the continuity of the limit function of the sequence $(\widetilde{u}_{k_n})_n$. Similarly one can prove that $\widetilde{t} > 0$.

Observe now that for a fixed $\xi \in (\widetilde{\tau}, \widetilde{t})$, we have

$$\widetilde{u}_{k_n}(\xi) = u_{k_n}(\xi) = u_0 + \int_0^\xi \varphi(u_{k_n}(t); k_n)\mathrm{d}t, \tag{4.48}$$

for n large enough; so from the dominated convergence theorem it follows that $(\widetilde{u}_{k_n})_n$ uniformly converges to u_{k_0} in $(\widetilde{\tau}, \widetilde{t})$.

If $\widetilde{\tau} > \xi_0$, then $\widetilde{u}_{k_n}(\tau_{k_n}) = v_0$ for every n sufficiently large; hence, by the above proved uniform convergence, we get $u_{k_0}(\widetilde{\tau}) = v_0$, a contradiction since u_{k_0} is strictly decreasing and $u_{k_0}(\xi_0) = v_0$. Hence, we have $\widetilde{\tau} = \xi_0$ and similarly we can prove that $\widetilde{t} = \xi_1$. Therefore, we conclude that $(\widetilde{u}_{k_n}(\xi))_n = (u_{k_n}(\xi))_n$ uniformly converges to u_{k_0} in $[\xi_0, \xi_1]$.

Now we assume $k_0 = 0$ and let $\delta > 0$ be such that $\alpha(k) \in (v_0, 1)$ for $k \in (0, \delta]$. For $k \in [0, \delta]$ and $\xi \in [\xi_0, \xi_1]$, the profile $u_k(\xi)$ is the unique solution of the initial value problem (see Theorem 3.7):

$$u'(\xi) = \frac{z_k(u)}{D(u; k)},$$
$$u(0) = u_0. \tag{4.49}$$

According to Remark 4.4, $z_k(u)/D(u; k)$ converges to $z_0(u)/D(u; 0)$ as $k \to 0$ for all $u \in [v_1, v_0]$. Hence, $|z_k(u)/D(u; k)|$ is bounded for $(u, k) \in [v_1, v_0] \times [0, \delta]$, and the same reasoning as before can be repeated in order to prove the pointwise convergence of $u_k(\xi)$ to $u_0(\xi)$ in $[\xi_0, \xi_1]$. Similarly, when $k_0 = 1$, we take $\delta > 0$ such that $\alpha(k) \in (0, v_1)$ for $k \in [1-\delta, 1)$. Since $u_k(\xi)$ is again the unique solution of (4.49) for $k \in [1-\delta, 1]$ and $\xi \in [\xi_0, \xi_1]$, we obtain also the pointwise convergence of $u_k(\xi)$ to $u_1(\xi)$ in $[\xi_0, \xi_1]$.

The uniform convergence on the all real line follows from Lemma 4.5.

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